

A priori Smith fair extra practice

From Homework 8.

Contents

1 The exercises

2 The solutions

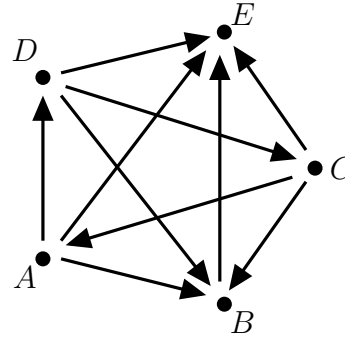
1 The exercises

1. Preference schedule is:

2	2	1	3	2
<i>A</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>B</i>	<i>B</i>	<i>C</i>	<i>A</i>	<i>C</i>
<i>D</i>	<i>C</i>	<i>A</i>	<i>B</i>	<i>A</i>
<i>C</i>	<i>D</i>	<i>D</i>	<i>D</i>	<i>B</i>

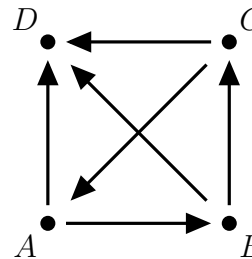
- (a) What is the Smith set?
 - (b) Which candidates have the power to be weak spoilers?
 - (c) Let $\Omega_P =$ plurality. Is D a weak spoiler for Ω_P ?
 - (d) Let $\Omega_{PC} =$ pairwise comparison. Is D a weak spoiler for Ω_{PC} ?
 - (e) Let $\Omega_R =$ Runoff. Is D a weak spoiler for Ω_R ?
 - (f) Let $\Omega_E =$ instant runoff/elimination. Is D a weak spoiler for Ω_E ?
 - (g) Let $\Omega_B =$ Borda count. Is D a weak spoiler for Ω_B ?
 - (h) Which WSM's can we conclude are NOT a priori Smith fair from these computations? Which WSM's do we still need to check?
 - (i) Now conduct a priori Smith fair WSM's on this example:
 - (a) What is the winner set for plurality?
 - (b) What is the winner set for a priori Smith fair plurality?
 - (c) What is the winner set for a priori Smith fair Borda Count?
 - (d) What is the winner set for a priori Smith fair Runoff?
 - (e) What is the winner set for a priori Smith fair Elimination?
 - (f) What is the winner set for pairwise comparison?
 - (g) What is the winner set for a priori pairwise comparison?
2. The next preference schedule:

1	2	1	1	2
A	A	C	D	B
D	D	B	E	C
B	C	A	C	E
C	E	E	A	A
E	B	D	B	D



- (a) What is the Smith set?
 - (b) Which candidates have the power to be weak spoilers?
 - (c) Check plurality for weak spoilers.
 - (d) Check runoff for weak spoilers.
 - (e) Check Instant runoff/Elimination for weak spoilers.
 - (f) Check Borda count for weak spoilers.
 - (g) Are there any weak spoilers for pairwise comparison?
3. The last preference schedule:

1	2	1	1	2	2
D	D	D	A	B	C
A	B	C	B	C	A
B	C	A	C	A	B
C	A	B	D	D	D



- (c) What is the Smith set?
- (d) What is the winner set for plurality?
- (e) What is the winner set for a priori Smith fair plurality?
- (f) What is the winner set for Borda Count?
- (g) What is the winner set for a priori Smith fair Borda Count?
- (h) What is the winner set for Elimination?
- (i) What is the winner set for a priori Smith fair Elimination?
- (j) What is the winner set for a priori Smith fair Runoff?
- (k) What is the winner set for pairwise comparison?
- (l) What is the winner set for a priori pairwise comparison?

Can we conclude any WSM's are NOT a priori Smith fair from these computations?

If YES, you should find the weak spoilers *in this election* for that WSM.

2 The solutions

1. Preference schedule is:

2	2	1	3	2
A	A	B	C	D
B	B	C	A	C
D	C	A	B	A
C	D	D	D	B

- (a) What is the Smith set? $S = \{A, B, C\}$
- (b) Which candidates have the power to be weak spoilers? D
- (c) Let $\Omega_P =$ plurality. Is D a weak spoiler for Ω_P ? First, $W_P = \{A\}$. When D is eliminated, $W_P = \{C\}$. The winner set changed, so D is indeed a weak spoiler for plurality.
- (d) Let $\Omega_{PC} =$ pairwise comparison. Is D a weak spoiler for Ω_{PC} ? No. One could check by computation, or just use that pairwise comparison is a priori Smith fair, which means it has no weak spoilers ever.
- (e) Let $\Omega_R =$ Runoff. Is D a weak spoiler for Ω_R ? First, the winner set is $W_R = \{C\}$ because A and C have the most first place votes, and when they enter the second runoff election, C wins. Then D is disqualified. Again, A and C have the most first place votes, and enter the second election, which C wins again. So $W'_R = \{C\}$ is the same as W_R , meaning D is NOT a weak spoiler for runoff.
- (f) Let $\Omega_E =$ instant runoff/elimination. Is D a weak spoiler for Ω_E ? First, C wins by elimination. In the first elimination round, B is removed with the fewest first place votes. In this second election, D has the fewest first place votes and is eliminated. Now it's just A vs. C , and C wins. So: $W_E = \{C\}$.

Now D is disqualified. We consider an election with only A , B , and C . With three candidates, runoff and instant runoff are the same, so we don't have to compute anything more to know that $W'_E = W'_R = \{C\}$ again. That means D is NOT a weak spoiler for elimination.

- (g) Let $\Omega_B =$ Borda count. Is D a weak spoiler for Ω_B ?

Compute the Borda scores for each candidate in this election to deduce that A wins by Borda count, ie- $W_B = \{A\}$.

$$\mathcal{B}(A) = 31$$

$$\mathcal{B}(B) = 24$$

$$\mathcal{B}(C) = 27$$

$$\mathcal{B}(D) = 18$$

(ps- we can check that this computation is right... $N = 10$ and $n = 4$, so the sum of the Borda scores should be $10 \times (4 + 3 + 2 + 1) = 100$.)

Now eliminate D and compute scores again. It would probably be convenient to write out the new preference schedule:

2	2	1	3	2
A	A	B	C	C
B	B	C	A	A
C	C	A	B	B

Borda scores are:

$$\mathcal{B}(A) = 23$$

$$\mathcal{B}(B) = 16$$

$$\mathcal{B}(C) = 21$$

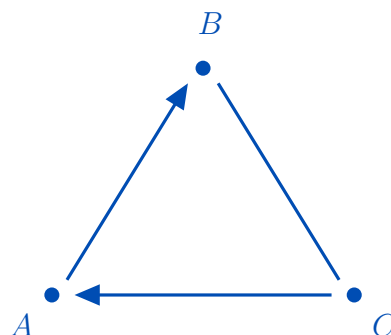
And A wins again by Borda count! So D is not a weak spoiler for Borda count.

- (h) Which WSM's can we conclude are NOT a priori Smith fair from these computations? Which WSM's do we still need to check? D is a weak spoiler for plurality, but not for anyone else. So plurality fails the a priori Smith fair criterion. We can't make any conclusions about the other WSM's from these computations!!!
- (i) Now compare the WSM's with their a priori Smith fair versions:

For our convenience, the preference schedule with the Smith set is copied and consolidated here:

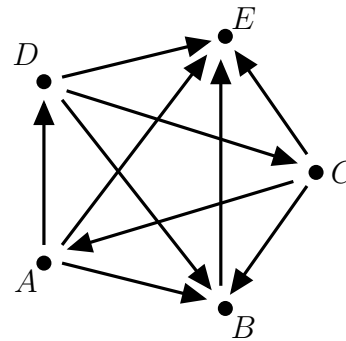
4	1	5
A	B	C
B	C	A
C	A	B

- (a) What is **was** the winner set for plurality **from before**? $W_P = \{D\}$.
- (b) What is the winner set for a priori Smith fair plurality? **If we do the plurality vote on only the Smith set, we see that C wins with 5 first place votes.**
- (c) What is the winner set for a priori Smith fair Borda Count? $\{A\}$
- (d) What is the winner set for a priori Smith fair Runoff? $\{C\}$
- (e) What is the winner set for a priori Smith fair Elimination? $\{C\}$
- (f) What is the winner set for pairwise comparison? $W_{PC} = \{C\}$
- (g) What is the winner set for a priori Smith fair pairwise comparison? $\{C\}$ (see graph below)



2. The next preference schedule:

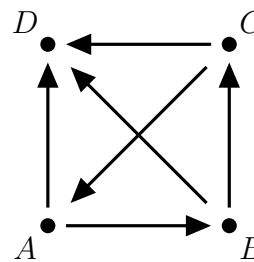
1	2	1	1	2
A	A	C	D	B
D	D	B	E	C
B	C	A	C	E
C	E	E	A	A
E	B	D	B	D



- (a) What is the Smith set? $S = \{A, D, C\}$
- (b) Which candidates have the power to be weak spoilers? E and B
- (c) Check plurality for weak spoilers. B is a weak spoiler
- (d) Check runoff for weak spoilers. B is a weak spoiler
- (e) Check Instant runoff/Elimination for weak spoilers. B is a weak spoiler
- (f) Check Borda count for weak spoilers. E is a weak spoiler. First A and C tie with a Borda score of 24. Without E , A wins alone with a Borda score of 20 (C 's new Borda score is 18). If B is removed then A and C again have the same Borda score of 20, so B is not a weak spoiler for Borda count.
- (g) Are there any weak spoilers for pairwise comparison? **neverrrr**
- (h) Which winner selection methods decidedly fail the a priori Smith fair criterion? **from this example: plurality, runoff, elimination, and Borda count!**

3. The last preference schedule:

1	2	1	1	2	2
D	D	D	A	B	C
A	B	C	B	C	A
B	C	A	C	A	B
C	A	B	D	D	D



- (c) What is the Smith set? $S = \{A, B, C\}$
- (d) What is the winner set for plurality? $W_P = \{D\}$
- (e) What is the winner set for a priori Smith fair plurality? $\{B\}$
- (f) What is the winner set for Borda Count?

$$\begin{aligned} \mathcal{B}(A) &= \mathcal{B}(D) = 21 \\ \mathcal{B}(B) &= \mathcal{B}(C) = 24 \end{aligned}$$

So the winner set is $\{B, C\}$.

- (g) What is the winner set for a priori Smith fair Borda Count?

$$\mathcal{B}(A) = 16$$

$$\mathcal{B}(B) = \mathcal{B}(C) = 19$$

So the winner set is again $\{B, C\}$.

- (h) What is the winner set for Elimination? $W_E = \{B\}$

- (i) What is the winner set for a priori Smith fair Elimination? $\{B\}$

- (j) ~~What is the winner set for Runoff?~~ B and C tie for number of first place votes after D , and then (if we just made up this tie breaker) they tie for number of second place votes, and number of third place votes, and number of last place votes, so this was not a well defined question at all and we should pretend it never happened.

- (k) What is the winner set for a priori Smith fair Runoff? $\{B\}$

- (l) What is the winner set for pairwise comparison? $\{A, B, C\}$

- (m) What is the winner set for a priori pairwise comparison? $\{A, B, C\}$

Can we conclude any WSM's are NOT a priori Smith fair from these computations? just plurality so far, but we already knew that it wasn't so this isn't too exciting.

Clearly D is a weak spoiler for plurality because $D \notin S$ wins by plurality.